

# On relation between discrete Frenet frames and the bi-Hamiltonian structure of the discrete nonlinear Schrödinger equation

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The discrete Frenet equation entails a local framing of a discrete, piecewise linear polygonal chain in terms of its bond and torsion angles. In particular, the tangent vector of a segment is akin the classical  $O(3)$  spin variable. Thus there is a relation to the lattice Heisenberg model, that can be used to model physical properties of the chain. On the other hand, the Heisenberg model is closely related to the discrete nonlinear Schrödinger (DNLS) equation. Here we apply these interrelations to develop a perspective on discrete chains dynamics: We employ the properties of a discrete chain in terms of a spinorial representation of the discrete Frenet equation, to introduce a bi-hamiltonian structure for the discrete nonlinear Schrödinger equation (DNLSE), which we then use to produce integrable chain dynamics.

## I. INTRODUCTION

String-like objects have important ramifications in many areas of physics, from understanding turbulent flows in fluid dynamics [1] to new forms of matter in high energy physics [2]. Similarly, piecewise linear discrete chains have many applications from the modelling of proteins in terms of its  $C\alpha$  backbone [3–5] to robotics and 3D virtual reality [6].

Here, the starting point is the observation by Hasimoto [7] that the one dimensional nonlinear Schrödinger equation (NLSE) describes string-like objects such as vortex filaments. For this, a change of variables is introduced that relates the NLSE to the Frenet frame representation of a space curve. The NLSE is a widely studied integrable model with various theoretical [8, 9] and practical [10] applications. Among the properties of the NLSE is the support of solitons as classical solutions. In the continuous case, the energy obtained from the hamiltonian of the NLSE relates to the geometry of a curve by a Hasimoto transformation. Thus, the corresponding solitons describe the buckling of the curve. Similarly, a properly discretized version of the NLSE preserves the integrability [11, 12] and supports soliton solutions. These can be utilized to model the buckling of piecewise linear polygonal chains, via a discretized version of the Hasimoto transformation [13].

In [13], a continuous string in  $\mathbb{R}^3$  is represented by a two component complex spinor. The dreibein Frenet equation becomes a two component spinor Frenet equation which relates to the integrable hierarchy of the NLSE. That way the conserved charges of the NLSE hierarchy can be considered as hamiltonians; and can be used to compute the energy and govern the time evolution of a string. Finally, it has been shown that the string hamiltonians in the NLSE hierarchy can be obtained alternatively by using a formalism of projection operators, originally introduced in the  $\mathbb{CP}^N$  models [17]. In particular, the connection between the projection operator formalism of the  $\mathbb{CP}^1$  model and the spinor Frenet equation has been explicitly presented.

In [13–15] the Frenet frame formalism of [13, 16] has been generalized to the case of polygonal strings that are piecewise linear. In particular, the Poisson geometry of a discrete string in  $\mathbb{R}^3$  has been derived. To do so, the Frenet frames are converted into a spinorial representation, the discrete spinor Frenet equation is interpreted in terms of a transfer matrix formalism, and the Poisson brackets are determined in terms of the spinor components.

The present article combines the results of [13] and [15] to study the discrete nonlinear Schrödinger equation (DNLSE), its mathematical structure and its use in modelling discrete, piecewise linear filamental chains and strings. In particular, a bi-hamiltonian description of the DNLSE, based on the discrete spinorial Frenet frame, is presented.

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## II. THE DISCRETE FRENET FRAME

Our starting point is the description of a discrete string in terms of an open and oriented, piecewise linear polygonal chain  $\vec{r}(s) \in \mathbb{R}^3$  [16]. The *arc length* parameter takes values on  $s \in [0, L]$  where  $L$  is the total length of the string. The vertices  $\mathcal{D}_i$  (for  $i \in \mathbb{Z}^+$ ) that specify the string are located at the points  $\vec{r}_i = (\vec{r}_0, \dots, \vec{r}_n)$  with  $\vec{r}(s_i) = \vec{r}_i$ ; while, the endpoints are  $\vec{r}(0) = \vec{r}_0$  and  $\vec{r}(L) = \vec{r}_n$ . In addition, the distance of the nearest neighbour vertices  $\mathcal{D}_i$  and  $\mathcal{D}_{i+1}$  is

$$|\vec{r}_{i+1} - \vec{r}_i| = s_{i+1} - s_i,$$

which are connected by the *line segments*

$$\vec{r}(s) = \frac{s - s_i}{s_{i+1} - s_i} \vec{r}_{i+1} - \frac{s - s_{i+1}}{s_{i+1} - s_i} \vec{r}_i, \quad s \in (s_i, s_{i+1}),$$

and are characterised by the unit length discrete tangent vector  $\vec{t}_i = (t_1, t_2, t_3)_i$  that points from  $\mathcal{D}_i$  to  $\mathcal{D}_{i+1}$

$$\vec{t}_i = \frac{\vec{r}_{i+1} - \vec{r}_i}{|\vec{r}_{i+1} - \vec{r}_i|}. \quad (1)$$

Then, the discrete Frenet frame at the vertex  $\mathcal{D}_i$  at the point  $\vec{r}_i$  can be introduced assuming that the vertices at the points  $\vec{r}_{i+1}$ ,  $\vec{r}_i$  and  $\vec{r}_{i-1}$  are not located on a common line (that is,  $\vec{t}_i$  and  $\vec{t}_{i-1}$  are not parallel): The unit binormal vector is defined by

$$\vec{b}_i = \frac{\vec{t}_{i-1} \times \vec{t}_i}{|\vec{t}_{i-1} \times \vec{t}_i|}$$

while the unit normal vector  $\vec{n}_i = \vec{b}_i \times \vec{t}_i$  takes the form

$$\vec{n}_i = \frac{-\vec{t}_{i-1} + (\vec{t}_{i-1} \cdot \vec{t}_i) \vec{t}_i}{|\vec{t}_{i-1} + (\vec{t}_{i-1} \cdot \vec{t}_i) \vec{t}_i|}.$$

The orthogonal triplet  $(\vec{t}, \vec{n}, \vec{b})_i$  constitutes the discrete Frenet frame associated the vertex  $\mathcal{D}_i$  for each  $i \in [1, n-1]$ . The aforementioned formulae are the two-point finite difference variants of the (corresponding) continuous Frenet frames. That way, the discretised curvature and torsion become geometrical. They correspond to the *bond* and *torsion angle* which define the transfer angle that maps the discrete Frenet frame at  $\mathcal{D}_i$  to  $\mathcal{D}_{i+1}$  through the *transfer matrix*  $\mathcal{R}_{i+1,i}$

$$\begin{aligned} \begin{pmatrix} \vec{n} \\ \vec{b} \\ \vec{t} \end{pmatrix}_{i+1} &= \mathcal{R}_{i+1,i} \begin{pmatrix} \vec{n} \\ \vec{b} \\ \vec{t} \end{pmatrix}_i \\ &= \begin{pmatrix} \cos \Theta \cos \Phi & \cos \Theta \sin \Phi & -\sin \Theta \\ -\sin \Phi & \cos \Phi & 0 \\ \sin \Theta \cos \Phi & \sin \Theta \sin \Phi & \cos \Theta \end{pmatrix}_{i+1} \begin{pmatrix} \vec{n} \\ \vec{b} \\ \vec{t} \end{pmatrix}_i. \end{aligned} \quad (2)$$

Note that, the transfer matrix is parametrized in terms of the two Euler angles  $(\Theta, \Phi)_i$  and represents an orthogonal rotation between the two consecutive discrete Frenet frames; i.e., is an element of  $SO(3)$  as illustrated in Figure [1]. The third Euler angle has been excluded due to the orthogonality of  $\vec{b}_i$  with  $\vec{t}_{i-1}$ .

From the transfer matrix (2) all the discrete Frenet frames can be constructed since

$$\vec{t}_{i+1} = \sin \Theta_{i+1} \cos \Phi_{i+1} \vec{n}_i + \sin \Theta_{i+1} \sin \Phi_{i+1} \vec{b}_i + \cos \Theta_{i+1} \vec{t}_i$$

implying that

$$\begin{aligned} \vec{t}_{i+1} \cdot \vec{t}_i &= \cos \Theta_{i+1} \\ \vec{t}_{i+1} \cdot \vec{n}_i &= \sin \Theta_{i+1} \cos \Phi_{i+1} \\ \vec{t}_{i+1} \cdot \vec{b}_i &= \sin \Theta_{i+1} \sin \Phi_{i+1}. \end{aligned} \quad (3)$$

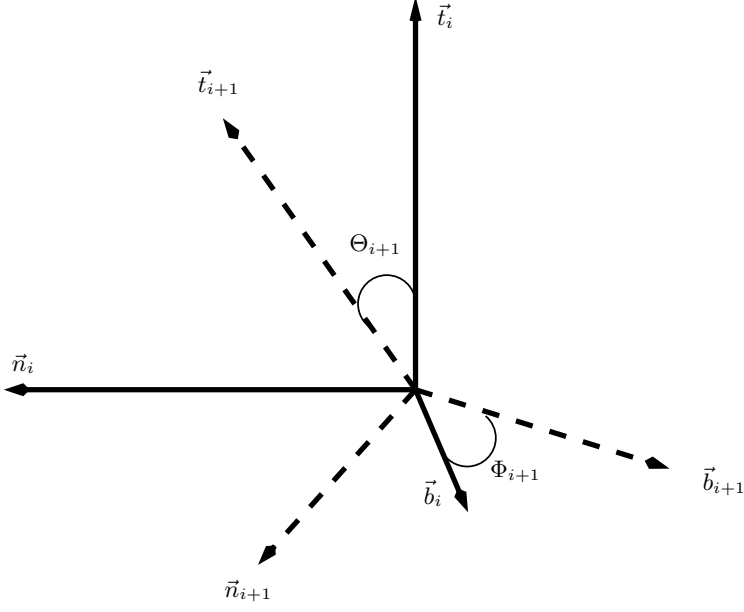


Figure 1: The bond and torsion angles of the transfer matrix  $\mathcal{R}_{i+1}$ .

Similarly, from the other rows one gets

$$\begin{aligned}\vec{b}_{i+1} \cdot \vec{b}_i &= \cos \Phi_{i+1} \\ \vec{b}_{i+1} \cdot \vec{n}_i &= -\sin \Phi_{i+1} \\ \vec{b}_{i+1} \cdot \vec{t}_i &= 0,\end{aligned}\tag{4}$$

and

$$\begin{aligned}\vec{n}_{i+1} \cdot \vec{t}_i &= -\sin \Theta_{i+1} \\ \vec{n}_{i+1} \cdot \vec{b}_i &= \cos \Theta_{i+1} \sin \Phi_{i+1} \\ \vec{n}_{i+1} \cdot \vec{n}_i &= \cos \Theta_{i+1} \cos \Phi_{i+1}.\end{aligned}\tag{5}$$

Thus,  $\Theta_{i+1}$  is the bond angle and  $\Phi_{i+1}$  is the torsion angle corresponding to the discrete analogue of the continuum curvature and torsion, respectively; i.e., the chain is determined by its bond and torsion angles uniquely up to global rotation and translation.

### III. THE DISCRETE SPINOR FRENET EQUATION

In [15], an alternative approach to represent discrete piecewise linear chains is introduced and developed to describe their time evolution, and the two dimensional Riemann surfaces that are swept by time evolution. In particular, the spinorial formulation of a discrete string in combination with the formalism of the discrete Frenet equations has been employed as described below.

Each link from vertex  $\mathcal{D}_i$  to vertex  $\mathcal{D}_{i+1}$  is associated to a two component *complex spinor*

$$\psi_i = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}_i,\tag{6}$$

where  $z_a^i$  (for  $a = 1, 2$ ) are complex variables with support on the *link*. The spinors are related to the unit length tangent vectors by the relation

$$\sqrt{g_i} \vec{t}_i = \langle \psi_i, \hat{\sigma} \psi_i \rangle$$

where  $\vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$  are the Pauli matrices and

$$\sqrt{g_i} \equiv |z_1^i|^2 + |z_2^i|^2$$

is a metric scale factor. Explicitly,

$$\begin{pmatrix} t_1^i + it_2^i \\ t_3^i \end{pmatrix} = \frac{1}{\sqrt{g_i}} \begin{pmatrix} 2\bar{z}_1^i z_2^i \\ |z_1^i|^2 - |z_2^i|^2 \end{pmatrix}. \quad (7)$$

Together with (1) this determines the spinor components  $z_\alpha^i$  in terms of the vertices  $\mathcal{D}_i$ , up to an overall phase. In addition, for each  $i$  the conjugate spinor  $\bar{\psi}_i$  is defined by introducing the *charge conjugation* operation  $\mathcal{C}$  that acts on  $\psi_i$  in the following way

$$\mathcal{C} \psi_i = -i\sigma_2 \psi_i^* = \bar{\psi}_i = \begin{pmatrix} -\bar{z}_2 \\ \bar{z}_1 \end{pmatrix}_i. \quad (8)$$

Since  $\mathcal{C}^2 = -\mathbb{I}$  the spinors are orthogonal by construction, i.e.  $\langle \psi_i, \bar{\psi}_i \rangle = 0$ . In addition, they are combined through the  $SU(2)$  matrix

$$u_i = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}_i$$

since

$$\psi_i = u_i \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \& \quad \bar{\psi}_i = u_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For completeness, note that

$$u_i \sigma_3 u_i^{-1} = (\vec{t} \cdot \vec{\sigma})_i$$

which is in agreement with equation (7).

In the continuum case, the parametrisation of the *link variables*  $z_\alpha^i$  in terms of the local angular variables  $\vartheta$  and  $\varphi_\pm = \varphi_2 \pm \varphi_1$  gives rise to the classical version of  $sl_2$  which appears in the case of local spin chains [13]. Its discrete analogue is of the form

$$z_1^i = \cos \frac{\vartheta}{2} e^{-\frac{i\varphi}{2}} |_i \quad \& \quad z_2^i = \sin \frac{\vartheta}{2} e^{\frac{i\varphi}{2}} |_i \quad (9)$$

assuming that  $\varphi_+ = 0$  and  $\varphi_- \equiv \varphi$  (due to the overall phase). Note that, in the local coordinate representation (9) the metric equals one.

In [15], it was shown that by combining the two spinors (6) and (8) into a Majorana spinor

$$\Psi_i = \begin{pmatrix} -\bar{\psi} \\ \psi \end{pmatrix}_i,$$

the discrete Frenet equation (2) is equivalent to the spinorial one

$$\Psi_{i+1} = \mathcal{U}_{i+1}^\dagger \Psi_i. \quad (10)$$

$\mathcal{U}_i$  is the *ensuing transfer matrix* in the chain of spinors  $\Psi_i$  defined by the *rotational variables*  $Z_a^i$  (for  $a = 1, 2$ )

$$\mathcal{U}_i = \begin{pmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix}_i$$

where

$$Z_1^i = \cos \frac{\Theta_i}{2} e^{-\frac{i\Phi_i}{2}} \quad \& \quad Z_2^i = \sin \frac{\Theta_i}{2} e^{\frac{i\Phi_i}{2}}. \quad (11)$$

It is straightforward to see that

$$\cos \Theta_i = (2|Z_2^i|^2 - 1) \quad \& \quad \cos \Phi_i = \frac{1}{2} \{ [(Z_1^i)^2 + (\bar{Z}_2^i)^2] + \text{cc} \}$$

in accordance with the Frenet frame obtained from equations (3) and (4). For example,

$$\vec{t}_{i+1} \cdot \vec{t}_i = (2|Z_2^i|^2 - 1)$$

and so on.

Finally, the relation between the link  $z_a^i$  and the rotational variables  $Z_a^i$  is given by (10) since

$$\mathcal{U}_{i+1}^\dagger = \frac{1}{\sqrt{g_i}} \left( \Psi_{i+1} \Psi_i^\dagger \right)$$

leading to

$$\begin{aligned} Z_1^i &= \frac{1}{\sqrt{g_i}} (\bar{z}_1^{i+1} z_1^i + \bar{z}_2^{i+1} z_2^i) \\ Z_2^i &= \frac{1}{\sqrt{g_i}} (z_1^{i+1} z_2^i - z_2^{i+1} z_1^i). \end{aligned} \quad (12)$$

Let us conclude, by discussing the discrete analogue of the equivalence between the projection operator formalism of the  $\mathbb{CP}^1$  model and the classic Frenet equation of a continuous string [13]. To do so, let us define the discrete projection operator

$$\mathbb{P}_i = \psi_i \otimes \psi_i^\dagger = u_i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u_i^{-1} \quad (13)$$

accompanied by the complemental projection operator

$$\bar{\mathbb{P}}_i = \bar{\psi}_i \otimes \bar{\psi}_i^\dagger = u_i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u_i^{-1} \quad (14)$$

such that the following relations exist

$$\mathbb{P}_i + \bar{\mathbb{P}}_i = \mathbb{I}_i \quad \& \quad \mathbb{P}_i^2 = \mathbb{P}_i$$

and so forth; for details, see [13]. Note that, the identity

$$\mathbb{P}_i^2 = \mathbb{P}_i$$

sets the metric equal to one; i.e.,  $\sqrt{g_i} = 1$  in accordance with the local angular parametrization (9). This was not the case in [15] where the Poisson algebra of the link  $z_\alpha^i$  and rotational  $Z_\alpha^i$  variables have been presented due to a different (than the projection operator formalism) approach. Therefore, the Poisson brackets of the corresponding link variables  $z_\alpha^i$  (given in Appendix B) are different from the ones presented in [15].

Finally, using the fact that the elements of the Frenet frame are  $su(2)$  Lie algebra generators the following equivalence has been derived

$$\mathbb{P}_i - \bar{\mathbb{P}}_i \simeq \vec{t}_i. \quad (15)$$

In the continuum [13], it was shown that the time evolution of the spinor (that describes the string) can be obtained from the integrable hierarchy of NLSE. In addition, it was shown that the conserved charges of the NLSE hierarchy can be expressed in terms of the projection operator formalism and, in particular, that are equivalent to the energy density of the  $\mathbb{CP}^1$  chiral model.

In the next section, we use the DSLNE to introduce an energy function for discrete polygonal chains. Equation (10) implies that the rotational angles are the natural variables for constructing energy functions for discrete piecewise linear strings. In analogy, with the continuum case, the energy must remain invariant under the local  $SO(2)$  frame rotation. For this we simply identify the spin variables with the tangent vector in accordance with (14). Thus, the energy function determines a chain dynamics which is manifestly invariant under the local frame rotation.

#### IV. BI-HAMILTONIAN STRUCTURE

In order to study the dynamics of a curve, Hamilton's equation of motion is a natural choice. In [13], using the projection formalism of the  $\mathbb{CP}^1$  model, it was shown that the string hamiltonians in the NLSE hierarchy are related to the spinor Frenet equation.

In fact, it was shown that the continuous string hamiltonian via the projection operator formalism

$$\mathcal{H}_{\text{NLSE}}^1 = \text{tr} (\mathbb{P}_s \mathbb{P}_s) \quad (16)$$

due to (15), is equivalent to the number operator in the NLSE hierarchy (otherwise, the Heisenberg spin chain hamiltonian)

$$\mathcal{H}_{\text{NLSE}}^2 = \frac{1}{2} (\partial_s \vec{t} \cdot \partial_s \vec{t}). \quad (17)$$

That is,

$$\mathcal{H}_{\text{NLSE}} = \mathcal{H}_{\text{NLSE}}^1 = \mathcal{H}_{\text{NLSE}}^2 = \frac{1}{2} |\kappa_c|^2, \quad (18)$$

where  $\kappa_c$  is the complex valued curvature (for details, see [13]).

In the discrete case, the polygonal curve is a set of sequentially connected segments whose length here is considered to be locally inextensible and therefore, the dynamics of the curve is essential the rotation of tangent vectors. Since the degree of freedom for tangent vectors is the angular rotation, the chain can be identified as a spin chain described by the lattice isotropic Heisenberg magnet model.

Thus, for discretising  $\mathcal{H}_{\text{NLSE}}$  given by (18) the forward finite difference scheme (i.e.,  $\partial_s f = f_{i+1} - f_i$ ) is applied. That way, one obtains

$$\mathcal{H}_{\text{DNLS}} = \mathcal{H}_{\text{DNLSE}}^1 = \mathcal{H}_{\text{DNLSE}}^2. \quad (19)$$

The first hamiltonian  $\mathcal{H}_{\text{DNLSE}}^1$  is the discretized version of the  $\mathbb{CP}^1$  sigma model (16) given by

$$\begin{aligned} \mathcal{H}_{\text{DNLSE}}^1 &= \sum_i \text{tr} (\mathbb{P}_{i+1} - \mathbb{P}_i)^2 \\ &= 2 \sum_i |z_1^i z_2^{i+1} - z_1^{i+1} z_2^i|^2 \end{aligned} \quad (20)$$

provided that  $\sqrt{g_i} = 1$ .

The second hamiltonian  $\mathcal{H}_{\text{DNLSE}}^2$  is the discretized version of NLSE hierarchy (17) given by

$$\begin{aligned} \mathcal{H}_{\text{DNLSE}}^2 &= \sum_i (1 - \vec{t}_{i+1} \cdot \vec{t}_i) \\ &= 2 \sum_i |Z_2^i|^2 \end{aligned} \quad (21)$$

that is, the isotropic (XXX) Heisenberg model hamiltonian which describes first neighbour spin-spin interactions. Note that, equation (19) holds due to (12).

Finally, the equation of motion is given by

$$\begin{aligned} \frac{d\vec{t}_i}{dt} &= \{\mathcal{H}_{\text{DNLS}}, \vec{t}_i\} \\ &= \vec{t}_i \times \frac{\partial \mathcal{H}_{\text{DNLS}}}{\partial \vec{t}_i}. \end{aligned}$$

In what follows, two a priori different discrete versions of the continuum NLSE are presented. Recall that, the NLSE equation is known to be integrable at the finite segment  $[0, L]$  when appropriate boundary conditions are imposed. The aforementioned integrability is extended in the discretized version since a bi-hamiltonian form of (20) is presented.

The first hamiltonian (20) is given in terms of the local angular variables (9), that is,

$$\mathcal{H}_{\text{DNLS}}^1 = \sum_i [1 - \cos \vartheta_{i+1} \cos \vartheta_i - \sin \vartheta_{i+1} \sin \vartheta_i \cos(\varphi_{i+1} - \varphi_i)] \quad (22)$$

associated with the only *non-zero* Poisson brackets:

$$\{\cos \vartheta_i, \varphi_j\} = \delta_{ij}. \quad (23)$$

In terms of the tangent vectors  $\vec{t}_i$ , the Poisson brackets are given by  $\{t_i^a, t_j^b\} = -\varepsilon^{abc} t_i^c \delta_{ij}$ . The consistency of the Poisson brackets between the two systems of variables is discussed in Appendix A.

Next, the second hamiltonian (21) in terms of the discrete Euler's angles (11) is presented. However, in order to obtain the corresponding Poisson brackets the relation between the two angular systems need to be known explicitly. In particular, from (12) and (9) it is straightforward to obtain that

$$\begin{aligned}\cos \Theta_{i+1} &= (2|Z_1^{i+1}|^2 - 1) \\ &= \cos \vartheta_{i+1} \cos \vartheta_i + \sin \vartheta_{i+1} \sin \vartheta_i \cos (\varphi_{i+1} - \varphi_i)\end{aligned}\quad (24)$$

$$\begin{aligned}\sin \Theta_{i+1} &= 2Z_1^{i+1} Z_2^{i+1} \\ &= \sin \vartheta_{i+1} \cos \vartheta_i - \cos \vartheta_{i+1} \sin \vartheta_i \cos (\varphi_{i+1} - \varphi_i)\end{aligned}\quad (25)$$

$$\begin{aligned}\cos \Phi_{i+1} &= \frac{1}{2} \{ [(Z_1^{i+1})^2 + (\bar{Z}_2^{i+1})^2] + \text{cc} \} \\ &= \cos (\varphi_{i+1} - \varphi_i)\end{aligned}\quad (26)$$

$$\begin{aligned}\sin \Phi_{i+1} &= \frac{i}{2} \{ [(Z_1^{i+1})^2 + (\bar{Z}_2^{i+1})^2] - \text{cc} \} \\ &= \cos \vartheta_i \sin (\varphi_{i+1} - \varphi_i),\end{aligned}\quad (27)$$

associated with the mixed terms

$$\begin{aligned}\cos \Theta_{i+1} \cos \Phi_{i+1} &= \frac{1}{2} \{ [(Z_1^{i+1})^2 - (\bar{Z}_2^{i+1})^2] + \text{cc} \} \\ &= \cos \vartheta_{i+1} \cos \vartheta_i \cos (\varphi_{i+1} - \varphi_i) + \sin \vartheta_{i+1} \sin \vartheta_i\end{aligned}\quad (28)$$

$$\begin{aligned}\cos \Theta_{i+1} \sin \Phi_{i+1} &= \frac{i}{2} \{ [(Z_1^{i+1})^2 - (\bar{Z}_2^{i+1})^2] - \text{cc} \} \\ &= \cos \vartheta_{i+1} \sin (\varphi_{i+1} - \varphi_i)\end{aligned}\quad (29)$$

$$\begin{aligned}\sin \Theta_{i+1} \cos \Phi_{i+1} &= Z_1^{i+1} \bar{Z}_2^{i+1} + \text{cc} \\ &= -\cos \vartheta_{i+1} \sin \vartheta_i + \sin \vartheta_{i+1} \cos \vartheta_i \cos (\varphi_{i+1} - \varphi_i)\end{aligned}\quad (30)$$

$$\begin{aligned}\sin \Theta_{i+1} \sin \Phi_{i+1} &= i (Z_1^{i+1} \bar{Z}_2^{i+1} - \text{cc}) \\ &= \sin \vartheta_{i+1} \sin (\varphi_{i+1} - \varphi_i).\end{aligned}\quad (31)$$

Also, due to consistency, the following condition is satisfied

$$\sin \vartheta_i \sin (\varphi_{i+1} - \varphi_i) = 0 \quad (32)$$

which follows from the trigonometric identity  $\sin^2 \Phi_{i+1} + \cos^2 \Phi_{i+1} = 1$  using equations (24) and (26). Also, from (12), (32) and (A1), it is straightforward to show the equivalence between the parametrization (3)-(5) and the equations (24)-(31).

So, in terms of the discrete Euler's angles, the second hamiltonian (21) takes the simple form

$$\mathcal{H}_{\text{DNLS}}^2 = \sum_i (1 - \cos \Theta_{i+1}). \quad (33)$$

associated with the Poisson brackets

$$\{\Theta_{i+1}, \Theta_i\} = -\sin \Phi_{i+1} \quad (34)$$

$$\{\Phi_{i+1}, \Phi_i\} = \frac{1}{\sin \Theta_i} (\sin \Phi_{i+1} \cot \Theta_{i+1} + \sin \Phi_i \cot \Theta_{i-1}) \quad (35)$$

$$\{\Theta_{i+1}, \Phi_i\} = \frac{\cos \Phi_{i+1}}{\sin \Theta_i} \quad (36)$$

$$\{\Theta_i, \Phi_i\} = -\cot \frac{\Theta_i}{2} - \cos \Phi_i \cot \Theta_{i-1} \quad (37)$$

$$\{\Theta_{i-1}, \Phi_i\} = \cot \frac{\Theta_{i-1}}{2} + \cos \Phi_i \cot \Theta_i \quad (38)$$

$$\{\Theta_{i-2}, \Phi_i\} = -\frac{\cos \Phi_{i-1}}{\sin \Theta_{i-1}} \quad (39)$$

in addition with the shifted ones to the left and right; i.e.,  $i \rightarrow i - 1$  and  $i \rightarrow i + 1$  respectively. For completeness, in Appendix C the consistence of the two algebras (23) and (34)-(39) is presented. Note that, the Poisson brackets (34)-(39) was first introduced in [14] but in another context.

Let us conclude by stating that two different hamiltonians for the DNLS have been obtained. The first  $\mathcal{H}_{\text{DNLS}}^1$  (22) has a "complicate" form associated by a "simple algebra" (23) while the second hamiltonian  $\mathcal{H}_{\text{DNLS}}^2$  (33) has a "simple" form associated with a "complicate" algebra (34)-(39).

## V. CONCLUSION

In summary, we have addressed the problem how to utilize extrinsic geometry in order to derive hamiltonian energy functions for discrete strings that move in three dimensional space. We utilize the transfer matrix formalism to consecutively map the discrete Frenet frame from one vertex of discrete curve to its neighbour. In this manner we arrive naturally to energy densities that relate to the integrable hierarchy of the NLSE defined on the lattice. In particular, under this frame the tangent vectors are considered to be classical spins and thus, the lattice Heisenberg model comes to be a natural choice of the chain's hamiltonian.

We have shown that the introduced discrete version of the NLSE is integrable since it holds a bi-hamiltonian structure. That way, the equation of motion of the tangent vectors and bond and torsion angles Poisson brackets are constructed and therefore, the dynamics of the lattice chain can be investigated. We will report on these issues in a future publication.

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## Appendix A

In the discrete case, the unit length tangent vector  $\vec{t} = \frac{1}{\sqrt{g_i}} (\psi^\dagger \vec{\sigma} \psi)$  and the binormal and normal vectors  $\vec{n} + i\vec{b} = \frac{1}{\sqrt{g_i}} (\psi^\dagger \vec{\sigma} \psi)$  in terms of the bond and torsion angles (9) are given by

$$\vec{t}_i = \frac{1}{\sqrt{g_i}} \begin{pmatrix} \bar{z}_1 z_2 + \bar{z}_2 z_1 \\ i(\bar{z}_2 z_1 - \bar{z}_1 z_2) \\ |z_1|^2 - |z_2|^2 \end{pmatrix}_i = \begin{pmatrix} \sin \vartheta \cos \varphi \\ \sin \vartheta \sin \varphi \\ \cos \vartheta \end{pmatrix}_i, \quad \vec{n}_{i+1} = \begin{pmatrix} \cos \vartheta \cos \varphi \\ \cos \vartheta \sin \varphi \\ -\sin \vartheta \end{pmatrix}_{i+1}, \quad \vec{b}_{i+1} = \begin{pmatrix} -\sin \varphi \\ \cos \varphi \\ 0 \end{pmatrix}_{i+1}, \quad (\text{A1})$$

where  $\psi$  is given by (6) and  $\sqrt{g_i} = 1$ . Then the continuum hamiltonian (18), using the forward finite difference scheme, becomes

$$\mathcal{H}_{\text{NLSE}} = 2 \sum_i (1 - \vec{t}_{i+1} \cdot \vec{t}_i) \quad (\text{A2})$$

accompanied by the Poisson bracket

$$\{t_i^a, t_j^b\} = -\varepsilon^{abc} t_i^c \delta_{ij} \quad (\text{A3})$$

in consistence with (20).

Another way to obtain (A3) is to use the continuum XXX chain formulation. That is, assume that the canonical coordinates are  $p_i = \cos \vartheta_i$  and  $q_i = \varphi_i$  where  $\{p_i, q_j\} = \delta_{ij}$  and  $\{q_i, q_j\} = 0 = \{p_i, p_j\}$  (in consistence with (23)). Also, assume that  $t_i^\pm = t_i^1 \pm it_i^2 = \sin \vartheta_i e^{\pm i\varphi_i}$  and  $t_i^z = t_i^3 = \cos \vartheta_i$  due to (A1). Thus, in terms of the canonical coordinates this implies that  $t_i^\pm = \sqrt{1 - p_i^2} e^{\pm i\varphi_i}$  and  $t_i^z = p_i$  and so it is easy to obtain that

$$\{t_i^+, t_j^-\} = -2it_i^z \delta_{ij}, \quad \{t_i^z, t_j^\pm\} = \mp i t_i^\pm \delta_{ij} \quad (\text{A4})$$



that is, (A3).

### Appendix B

In terms of the link variables  $z_a^i$  variables the associated Poisson brackets are non-trivial. In fact, from (9) using (23) it can be shown that the only non-vanishing Poisson brackets are the following:

$$\begin{aligned}\{z_a^i, \bar{z}_a^j\} &= \frac{i}{4} \delta_{ij}, \\ \{z_1^i, z_2^j\} &= -\frac{i}{8} \left( \frac{z_1^i}{z_2^i} - \frac{z_2^i}{z_1^i} \right) \delta_{ij} \\ \{z_1^i, \bar{z}_2^j\} &= -\frac{i}{8} \left( \frac{z_1^i}{z_2^i} + \frac{\bar{z}_2^i}{\bar{z}_1^i} \right) \delta_{ij}.\end{aligned}$$

Recall that,  $|z_1^i|^2 + |z_2^i|^2 = 1$ .

### Appendix C

Let us conclude by presented some explicit examples of how the Poisson brackets of the Euler's angles  $(\Theta, \Phi)_i$  can be obtained directly from the corresponding ones of the local angular variables  $(\vartheta, \varphi)_i$ .

Let us start by considering the first bracket (34). Substituting equation (24) in the right hand side of (34) and applying the properties of the Poisson bracket one gets

$$\begin{aligned}\{\cos \Theta_{i+1}, \cos \Theta_i\} &= \frac{\partial}{\partial \vartheta_i} (\cos \Theta_{i+1}) \{\vartheta_i, \varphi_i\} \frac{\partial}{\partial \varphi_i} (\cos \Theta_i) - \frac{\partial}{\partial \varphi_i} (\cos \Theta_{i+1}) \{\vartheta_i, \varphi_i\} \frac{\partial}{\partial \vartheta_i} (\cos \Theta_i) \\ &= \sin \Theta_{i+1} \cos \Phi_{i+1} \left( \frac{-1}{\sin \vartheta_i} \right) [-\sin \vartheta_i \sin \vartheta_{i-1} \cos(\varphi_i - \varphi_{i-1})] \\ &\quad - \sin \vartheta_{i+1} \sin \vartheta_i \sin(\varphi_{i+1} - \varphi_i) \left( \frac{-1}{\sin \vartheta_i} \right) (-\sin \Theta_i) \\ &= 0 - \sin \vartheta_{i+1} \sin(\varphi_{i+1} - \varphi_i) \sin \Theta_i \\ &= -\sin \Theta_{i+1} \sin \Phi_{i+1} \sin \Theta_i.\end{aligned}\tag{C1}$$

Note that, the condition (32) and the equation (31) is (also) been used.

However, the right-hand side of the above equation gives

$$\{\cos \Theta_{i+1}, \cos \Theta_i\} = \sin \Theta_{i+1} \sin \Theta_i \{\Theta_{i+1}, \Theta_i\}.\tag{C2}$$

Thus, equation (34) is obtained.

For completeness, let us (also) show how the Poisson bracket (36) can be derived. Substituting equation (24) and (27) in the right hand side of (36) and applying the properties of the Poisson bracket one gets

$$\begin{aligned}\{\cos \Theta_{i+1}, \sin \Phi_i\} &= \frac{\partial}{\partial \vartheta_i} (\cos \Theta_{i+1}) \{\vartheta_i, \varphi_i\} \frac{\partial}{\partial \varphi_i} (\sin \Phi_i) - \frac{\partial}{\partial \varphi_i} (\cos \Theta_{i+1}) \{\vartheta_i, \varphi_i\} \frac{\partial}{\partial \vartheta_i} (\sin \Phi_i) \\ &= \sin \Theta_{i+1} \cos \Phi_{i+1} \left( \frac{-1}{\sin \vartheta_i} \right) \cos \vartheta_{i-1} \cos(\varphi_i - \varphi_{i-1}) \\ &= \sin \Theta_{i+1} \cos \Phi_{i+1} \left( \frac{-1}{\sin \vartheta_i} \right) \cos \vartheta_{i-1} \cos \Phi_i \\ &= \sin \Theta_{i+1} \cos \Phi_{i+1} \sin \Theta_i \cos \Phi_i.\end{aligned}\tag{C3}$$

Equation (26) has (also) been used as well as the identity  $\sin \Theta_i \equiv \frac{\sin \vartheta_i}{\cos \vartheta_{i-1}}$  obtained when (31) is divided by (27).

However, the right-hand side of the above equation gives

$$\{\cos \Theta_{i+1}, \sin \Phi_i\} = -\sin \Theta_{i+1} \cos \Phi_i \{\Theta_{i+1}, \Phi_i\}.\tag{C4}$$

Therefore, equation (36) is derived.

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